

Čerenkov radiation by charged particles in an external gravitational field

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Abstract

Charged particles in the geodesic trajectory of an external gravitational field do not emit electromagnetic radiation. This is expected from the application of the equivalence principle. We show here that charged particles propagating in an external gravitational field with non-zero components of the Ricci tensor can emit radiation by the Čerenkov process. The external gravitational field acts like an effective refractive index for light. Since the Ricci tensor cannot be eliminated by a change of coordinates, there is no violation of the equivalence principle in this process.

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An accelerated charge emits electromagnetic radiation at a rate prescribed by the well known Larmour formula [1]. When the acceleration is due to an external magnetic field then the resulting synchrotron radiation is the most important mode of energy loss of a charged particle in astrophysical situations and is observable as pulsar signals [2]. Radiation from accretion discs around neutron stars etc. is caused by collisional acceleration of the charged particles [3]. However when the acceleration is caused solely by an external gravitational field then there is no curvature radiation [4]. This result is consistent with the equivalence principle according to which, it is always possible to find local inertial frame where the acceleration due to gravity disappears. In this inertial frame, the particle has a constant velocity and will therefore not emit curvature radiation. A charged particle with a constant velocity however can emit Čerenkov radiation if there is an external medium such that the phase velocity of the photons in that medium is less than the velocity of the particle [5]. We show here that the background gravitational field has an effective refractive index given by $n_\gamma^2(k_0) = |g^{00}|(1 - R^i_i/|g^{00}|k_0^2)$, where the i summation is over the spatial components of the Ricci tensor R^μ_ν and k_0 is the photon frequency. If the quantity R^i_i is negative for a certain metric, then the effective gravitational refractive index of that metric $n_\gamma(k_0) > 1$ and radiation by the Čerenkov process is kinematically allowed. According to the equivalence principle it is possible to find an inertial frame at each point where the metric is flat. However the Riemann tensor which is a measure of the relative acceleration between two nearby geodesics [6] does not necessarily vanish in an inertial frame, and the components of the Ricci tensor and the curvature scalar are non-zero. Since the Čerenkov emission is proportional to the Ricci tensor, it is non-vanishing in the inertial frame of the particle unlike the usual curvature radiation. An example of a metric that has such a property is the gravitational field of the (dark + luminous) matter in our galaxy. The Newtonian potential which gives rise to the flat rotation curves is given by $\phi(r) = -v_c^2[1 - \ln(r/r_{max})]$, $r < r_{max} \approx 100$ kpc; the corresponding gravitational refractive index in the local inertial frame is $n_\gamma^2(k_0) = 1 + (v_c^2/r^2 k_0^2)[1 + 4 \ln(r/r_{max})]$, and therefore the necessary condition for Čerenkov radiation is satisfied. We show that for a charged fermion with four momentum p and mass

m , the rate of energy radiated by the Čerenkov process is

$$\frac{dE}{dt} = \frac{Q^2 \alpha_{em} (-R^{\hat{i}}_{\hat{i}})}{4\pi} \left[\frac{1}{2m^2} (-R^{\hat{i}}_{\hat{i}}) + \ln\left(\frac{2p_{\hat{0}}}{m}\right) - \frac{1}{m} (-R^{\hat{i}}_{\hat{i}})^{\frac{1}{2}} \right]. \quad (1)$$

where the hats over the indices represent the components in the local inertial reference frame. For a neutral fermion with mass m and a non-zero magnetic moment μ the rate of energy radiated by Čerenkov process is given by

$$\frac{dE}{dt} = \frac{\mu^2 (-R^{\hat{i}}_{\hat{i}})^2}{4\pi} \left[\ln\left(\frac{2p_{\hat{0}}}{m}\right) - \frac{(-R^{\hat{i}}_{\hat{i}})^{\frac{1}{2}}}{m} \right]. \quad (2)$$

Consider the photon emission by Čerenkov process $f(p) \rightarrow f(p') + \gamma(k)$ in the local inertial frame of the incoming fermion. The conservation of energy momentum *i.e.* $p^{\hat{\mu}} = p'^{\hat{\mu}} + k^{\hat{\mu}}$ implies that the Čerenkov radiation angle between \mathbf{p} and \mathbf{k} vectors are given by the relation

$$\cos \theta = \frac{g^{ij} p_i k_j}{(g^{ij} k_i k_j)^{\frac{1}{2}} (g^{lm} p_l p_m)^{\frac{1}{2}}} \equiv \frac{|g^{\hat{0}\hat{0}}| p_{\hat{0}} k_{\hat{0}}}{|\hat{\mathbf{p}}| |\hat{\mathbf{k}}|} \left(1 + \frac{k^2}{2|g^{\hat{0}\hat{0}}| p_{\hat{0}} k_{\hat{0}}} \right); \quad g^{\hat{\alpha}\hat{\beta}} = \text{diag}(-1, 1, 1, 1). \quad (3)$$

A necessary condition for the photon emission is therefore that the right hand side of the eqn. (3) is ≤ 1 or ≥ -1 . In order to see if this condition is satisfied we need the dispersion relations and the effective refractive indices of fermions $n_f \equiv |\mathbf{p}|/p_0$ and photons $n_\gamma \equiv |\mathbf{k}|/k_0$. In terms of the refractive indices the necessary condition for the Čerenkov process can be written as

$$-1 \leq \frac{|g^{\hat{0}\hat{0}}|}{n_\gamma n_f} \left[1 + (n_\gamma^2 - 1) \frac{k_{\hat{0}}}{2|g^{\hat{0}\hat{0}}| p_{\hat{0}}} \right] \leq 1. \quad (4)$$

For very high energy particles n_f can be arbitrarily close to one. Then the condition (4) gives the frequency range of the emitted photon by Čerenkov process.

The propagation of fermions in a curved background is governed by the Dirac equation

$$D_{\hat{\alpha}} \Psi = m \Psi, \quad (5)$$

where the covariant derivative $D_{\hat{\alpha}}$ is defined as [7]

$$D_{\hat{\alpha}} = e_{\hat{\alpha}}^{\mu} \left(\partial_{\mu} + \frac{i}{2} \omega_{\mu}^{\lambda\hat{\beta}} \sigma_{\lambda\hat{\beta}} \right), \quad (6)$$

where $e_{\hat{\alpha}}^{\mu}$ are the tetrads connecting the local inertial frame coordinates $x^{\hat{\alpha}}$ with the coordinate frame x_{μ} and $\sigma_{\hat{\alpha}\hat{\beta}} = \frac{i}{4}[\gamma_{\hat{\alpha}}, \gamma_{\hat{\beta}}]$. The spin connections $\omega_{\mu}^{\hat{\alpha}\hat{\beta}}$ can be expressed in terms of the tetrads as

$$\omega_{\mu}^{\hat{\alpha}\hat{\beta}} = e^{\hat{\alpha}\nu} e_{\nu}^{\hat{\beta}}{}_{;\mu} \quad . \quad (7)$$

From eqn. (5), we obtain the anticommutator relation

$$\frac{1}{2}\{\gamma^{\hat{\alpha}}D_{\hat{\alpha}}, \gamma^{\hat{\beta}}D_{\hat{\beta}}\}\Psi = m^2\Psi. \quad (8)$$

Using eqn. (6) and simplifying the anti commutator bracket in eqn. (8), we obtain the wave equation for the propagation of the fermions in a curved background

$$[g^{\mu\nu}\partial_{\mu}\partial_{\nu} - \frac{1}{4}R - m^2]\Psi = 0, \quad (9)$$

where we have used the identity [8]

$$R = R^{\hat{\alpha}\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}} = e^{\mu}{}_{\hat{\alpha}}e^{\nu}{}_{\hat{\beta}}[\partial_{\mu}\omega_{\nu}^{\hat{\alpha}\hat{\beta}} - \partial_{\nu}\omega_{\mu}^{\hat{\alpha}\hat{\beta}} - \omega_{\mu}^{\hat{\lambda}\hat{\alpha}}\omega_{\nu\hat{\lambda}}^{\hat{\beta}} + \omega_{\nu}^{\hat{\lambda}\hat{\alpha}}\omega_{\mu\hat{\lambda}}^{\hat{\beta}}], \quad (10)$$

for the curvature scalar in term of the spin connections and tetrads. The dispersion relation for the fermions is

$$g^{\mu\nu}p_{\mu}p_{\nu} + \frac{1}{4}R + m^2 = 0, \quad (11)$$

and the effective refractive index of the curved background is given by

$$n_f = \frac{|\mathbf{p}|}{p_0} = |g^{00}|^{\frac{1}{2}}(1 - \frac{m^2 + R/4}{|g^{00}|p_0^2})^{\frac{1}{2}}. \quad (12)$$

Turning to photons, the curved space lagrangian

$$L = \sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}, \quad (13)$$

gives the wave equation

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}A^{\alpha} - R^{\alpha}{}_{\mu}A^{\mu} = 0, \quad (14)$$

after imposing the gauge condition

$$\nabla_\mu A^\mu = 0. \quad (15)$$

In the eikonal approximation the corresponding wave equation is given by

$$g^{\mu\nu} k_\mu k_\nu A^i + R^i{}_j A^j = 0, \quad (16)$$

and the dispersion relation is obtained from the condition

$$\det \begin{pmatrix} k^\mu k_\mu + R^1{}_1 & R^1{}_2 & R^1{}_3 \\ R^2{}_1 & k^\mu k_\mu + R^2{}_2 & R^2{}_3 \\ R^3{}_1 & R^3{}_2 & k^\mu k_\mu + R^3{}_3 \end{pmatrix} = 0. \quad (17)$$

The dispersion relation in the leading order in Ricci tensor for a spherically symmetric metric is

$$g^{\mu\nu} k_\mu k_\nu + R^i{}_i = 0, \quad (18)$$

and the corresponding refractive index is given by

$$n_\gamma = \frac{|\mathbf{k}|}{k_0} = |g^{00}|^{\frac{1}{2}} \left(1 - \frac{R^i{}_i}{|g^{00}|k_0^2}\right)^{\frac{1}{2}}. \quad (19)$$

In order that $n_\gamma > 1$ we must have $R^i{}_i < 1$. Substituting eqn. (12) and eqn. (19) in eqn. (4), we find that the Čerenkov angle in a local inertial frame is given by

$$\cos \theta = \left[1 + \frac{R^i{}_i}{2k_0^2} \left(1 - \frac{k_{\hat{0}}}{p_{\hat{0}}}\right) + \frac{1}{2p_{\hat{0}}^2} \left(\frac{1}{4}R + m^2\right)\right]. \quad (20)$$

The rate of energy radiated as photons in the Čerenkov radiation process $f(p) \rightarrow f(p') + \gamma(k)$ in the local inertial frame of the incoming fermion is given by

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2p_{\hat{0}}} \int \frac{d^3\hat{k}}{2k_{\hat{0}}} \frac{d^3\hat{p}'}{2p'_{\hat{0}}} \frac{1}{(2\pi)^2} \delta^{(4)}(\hat{p} - \hat{p}' - \hat{k}) |\mathcal{M}|^2 k_{\hat{0}} \\ &= \frac{1}{2p_{\hat{0}}} \int \frac{d^3\hat{k}}{2k_{\hat{0}}} \frac{d^4\hat{p}'}{(2\pi)^2} \theta(p'_{\hat{0}}) \delta(\hat{p}'^2 - m^2) \delta^{(4)}(\hat{p} - \hat{p}' - \hat{k}) |\mathcal{M}|^2 k_{\hat{0}}, \end{aligned} \quad (21)$$

where for the fermion with charge Q ,

$$|\mathcal{M}|^2 = 4Q^2 \alpha_{em} \left[\frac{n_\gamma^2 - 2}{n_\gamma^2} (p'^{\hat{\alpha}} p_{\hat{\alpha}} - m^2) + \frac{2p'^{\hat{\alpha}} k_{\hat{\alpha}}}{n_\gamma^2 k_{\hat{0}}^2} (p_{\hat{0}} k_{\hat{0}} - p^{\hat{\alpha}} k_{\hat{\alpha}}) + \frac{2p^{\hat{\alpha}} k_{\hat{\alpha}}}{n_\gamma^2 k_{\hat{0}}} p'_{\hat{0}} + \frac{n_\gamma^2 - 1}{n_\gamma^2} p'_{\hat{0}} p_{\hat{0}} \right]. \quad (22)$$

Writing the mass-shell condition for the outgoing fermion as

$$\delta(p'^{\hat{\alpha}} p'_{\hat{\alpha}} - m^2) = \frac{1}{2|\hat{\mathbf{p}}||\hat{\mathbf{k}}|} \delta\left(\frac{k^2 - 2p^{\hat{0}}k_{\hat{0}}}{2|\hat{\mathbf{p}}||\hat{\mathbf{k}}|} - \cos\theta\right), \quad (23)$$

we have

$$\frac{dE}{dt} = \frac{Q^2 \alpha_{em}}{4\pi n_\gamma^2} \int \frac{\hat{\mathbf{k}}^2 d\hat{\mathbf{k}}}{p_{\hat{0}} k_{\hat{0}} |\hat{\mathbf{p}}||\hat{\mathbf{k}}|} \int_{-1}^1 d\cos\theta \delta\left(\frac{k^2 - 2p^{\hat{0}}k_{\hat{0}}}{2|\hat{\mathbf{p}}||\hat{\mathbf{k}}|} - \cos\theta\right) [p_{\hat{0}}(p_{\hat{0}} - k_{\hat{0}}) \frac{(n_\gamma^2 - 1)}{n_\gamma^2} + \frac{k^2}{2n_\gamma^2}] k_{\hat{0}}. \quad (24)$$

From eqn. (21) it is clear that the integral is non-zero when the criterion (20) is obeyed. The condition (20) therefore gives the range of the emitted photon frequency in terms of the incoming fermion energy and the refractive indices of the fermion and the photon in the external gravitational field. Performing the integral over the δ -function in (21) and substituting for k in terms of the refractive index n_γ , we have the expression for the energy radiated by a charged particle in a background gravitational field by the Čerenkov process given by

$$\frac{dE}{dt} = \frac{Q^2 \alpha_{em}}{4\pi p_{\hat{0}}^2} \int_{\omega_1}^{\omega_2} dk_{\hat{0}} (n_\gamma^2 - 1) \left[\frac{k_{\hat{0}}^2}{2} + p_{\hat{0}}(p_{\hat{0}} - k_{\hat{0}}) \right] k_{\hat{0}}, \quad (25)$$

where the range of $k_{\hat{0}}$ is where the criterion (20) is satisfied. The range of allowed photon frequency in local inertial frame (ω_1, ω_2) is obtained by using (12) and (19) in (4) as

$$\left(\frac{-R^{\hat{i}}_{\hat{i}}}{4}\right)^{\frac{1}{2}} \leq k_{\hat{0}} \leq (-R^{\hat{i}}_{\hat{i}})^{\frac{1}{2}} (p_{\hat{0}}/m). \quad (26)$$

From the expression for the refractive index (19) it is clear that the radiation rate given by (25) is positive for the background metrics for which $R_i^i < 0$. The rate of energy loss in this background through Čerenkov radiation is obtained by substituting (19) in (25) and using (26) to yield

$$\frac{dE}{dt} = \frac{Q^2 \alpha_{em} (-R^{\hat{i}}_{\hat{i}})}{4\pi} \left[\frac{1}{2m^2} (-R^{\hat{i}}_{\hat{i}}) + \ln\left(\frac{2p_{\hat{0}}}{m}\right) - \frac{1}{m} (-R^{\hat{i}}_{\hat{i}})^{\frac{1}{2}} \right]. \quad (27)$$

Neutral fermions with non-zero magnetic dipole moment like neutrons and possibly neutrinos can also emit Čerenkov radiation. The amplitude for the process $f(p) \rightarrow f(p') + \gamma(k)$ through the magnetic dipole vertex is given by

$$\mathcal{M} = \frac{\mu}{n_\gamma} \sigma^{\mu\nu} \epsilon_\mu k_\nu, \quad (28)$$

where μ is the magnetic dipole moment of f . Substitution of $|\mathcal{M}|^2$ in (21) followed by the integration over the relevant variables yields [9]

$$\frac{dE}{dt} = \frac{\mu_\nu^2}{16\pi p_0^2} \frac{(n_\gamma^2 - 1)^2}{n_\gamma^2} \int_{\omega_1}^{\omega_2} k_0 dk_0 [4p_0^2 k_0^2 - 4p_0 k_0^3 - (n_\gamma^2 - 1)k_0^4]. \quad (29)$$

Then we found out the rate of energy radiated in this case to be

$$\frac{dE}{dt} = \frac{\mu^2 (-R^i_i)}{4\pi} \left[\ln\left(\frac{2p_0}{m}\right) - \frac{(-R^i_i)^{\frac{1}{2}}}{m} \right]. \quad (30)$$

An example of a metric where the Ricci tensor R^i_i are negative is the gravitational field of the dark matter distribution in our galaxy. The galactic metric is described by the components $g_{rr} = (1 - 2\phi)$, $g_{tt} = (1 + 2\phi)$ where ϕ is the Newtonian potential that gives rise to the flat rotation curve and is given by

$$\begin{aligned} \phi(r) &= -v_c^2 \left[1 - \ln\left(\frac{r}{r_{max}}\right) \right], \quad r < r_{max} \\ &= -\frac{GM_g}{r}, \quad r > r_{max}, \end{aligned} \quad (31)$$

where $v_c \approx 220$ km/sec is the rotational velocity of the galactic halo, $r_{max} \approx 100$ kpc is the extent of the dark matter halo. The components of the Ricci tensor corresponding to the Newtonian potentials (28) are

$$\begin{aligned} R^t_t = R^r_r &= -\frac{v_c^2}{r^2}, \\ R^\theta_\theta = R^\phi_\phi &= -\frac{2v_c^2}{r^2} \ln\left(\frac{r}{r_{max}}\right), \end{aligned} \quad (32)$$

for $r < r_{max}$ and $R^\mu_\nu = 0$ for $r > r_{max}$. The refractive index of light in its inertial frame in such a gravitational field is given by

$$n_\gamma^2 = 1 + \frac{v_c^2}{r^2 k_0^2} [1 + 4 \ln(r/r_{max})], \quad (33)$$

which is greater than 1 as long as $\ln(r/r_{max}) > -0.25$. Then the rate of Čerenkov radiation is given by substituting (32) in (27) to yield

$$\frac{dE}{dt} = \frac{Q^2 \alpha_{em} v_c^2}{4\pi r^2} [1 + 4 \ln(r/r_{max})] \ln \frac{2p_{\hat{0}}}{m}. \quad (34)$$

The total energy loss for extra galactic cosmic rays due to the Čerenkov process by the time they arrive on the earth at distance r is given by

$$\Delta E(r) = \int_r^{r_{max}} \frac{dE}{dt} dr = \frac{Q^2 \alpha_{em} v_c^2}{4\pi} \ln\left(\frac{2p_{\hat{0}}}{m}\right) \left(\frac{1}{r} - \frac{1}{r_{max}}\right). \quad (35)$$

For cosmic ray protons of energy 10^{19} eV, this turns out to be 10^{-23} eV. The energy loss is expected to be large in the accretion discs of compact sources and may be of observational significance.

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